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SELF-SIMILAR SOLUTION OF THE ONE-DIMENSIONAL PROBLEM OF THERMOCAPILLARY MOTION OF AN EMULSION

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It is proved that the problem of one-dimensional motion of an emulsion under the action of thermocapillary forces has a self-similar solution in a semi-infinite interval. The behavior of the solution is illustrated by numerical examples for aluminum–lead emulsions, in which the carrier phase is lead or aluminum. The solution is compared with the solution of the self-similar problem linearized in the low impurity concentration.

Key words: *emulsion, thermocapillary forces, self-similar solution.*

1. Formulation of the Problem. In 1995, Voinov and Pukhnachev formulated a model for the motion of an emulsion in a field of microaccelerations and thermocapillary forces [1]. This model (the stability of simple solutions, the problem of emulsion solidification in linearized formulations, simple discontinuous solutions) has been the subject of a number of studies; a review of the main results is given in [2]. However, even in the one-dimensional case, the model is very complex and insufficiently studied. It is of interest to find and study nontrivial exact solutions of the complete problem, in particular, the self-similar solution with the variable $\xi = x/\sqrt{t}$. We note that a self-similar solution of the linearized problem of emulsion solidification is considered in [3].

The constitutive equations of the Voinov–Pukhnachev model [1] are written as

$$\frac{\partial c}{\partial t} + \operatorname{div}(c\boldsymbol{u}) = 0,$$

$$\frac{\partial(1-c)}{\partial t} + \operatorname{div}((1-c)\boldsymbol{v}) = 0,$$

$$\rho_d c\Big(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) + \rho_m(1-c)\Big(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\Big)$$

$$= -\nabla p + \operatorname{div}(\mu_m(1+cN)(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^*)) + \rho_d c\boldsymbol{g} + \rho_m(1-c)\boldsymbol{g},$$

$$\rho_d\lambda_d c\Big(\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T\Big) + \rho_m\lambda_m(1-c)\Big(\frac{\partial T}{\partial t} + \boldsymbol{v}\cdot\nabla T\Big) = \operatorname{div}(k(c)\nabla T);$$

$$\boldsymbol{u} - \boldsymbol{v} = K\boldsymbol{g} + L\nabla T. \tag{1.1}$$

Here $c \ll 1$ is the dispersed-phase concentration, T is the total temperature, $\boldsymbol{u}$ and $\boldsymbol{v}$ are the averaged velocities of the dispersed and carrier phases, respectively, p is the pressure, ρ is the density, μ is the dynamic viscosity, λ is the specific heat, k is the thermal conductivity, and

$$N = \frac{\mu_m + 5\mu_d/2}{\mu_m + \mu_d}, \qquad K = \frac{2R^2(\rho_d - \rho_m)(\mu_m + \mu_d)}{3\mu_m(2\mu_m + 3\mu_d)}, \qquad L = \frac{2Rk_m\sigma_T}{(2\mu_m + 3\mu_d)(2k_m + k_d)},$$

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where R is the radius of spherical inclusions and σ_T is the derivative (with the inverse sign) of the surface tension with respect to temperature; the subscripts d and m denote the parameters of the dispersed and carrier phases, respectively.

The thermal conductivity of the mixture was determined using a linear approximation (with respect to small values of c) of the Maxwell formula. Since the model of [1] was constructed under the assumption of a low concentration of the dispersed phase, the use of the linear approximation is reasonable and justified. In the case considered, it is more convenient to use nonlinear thermal conductivity $k(c)$. Its value is bounded from below and from above: $\min(k_d, k_m)$ and $\max(k_d, k_m)$. Let $\max|k'(c)| \le P$. For simplicity in obtaining estimates, we assume that $P/\min(k_d, k_m) \le 5$ although the Maxwell formula yields a more accurate estimate.

In the case of one-dimensional motion with plane waves, the problem reduces to the system of two equations [2]

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}\Big\{\Big[c\Big(Kg + L\,\frac{\partial T}{\partial x}\Big) - f\Big](1-c)\Big\} = 0,$$

$$\Big[\rho_d\lambda_d c + \rho_m\lambda_m(1-c)\Big]\Big(\frac{\partial T}{\partial t} + f\,\frac{\partial T}{\partial x}\Big) + (\rho_d\lambda_d - \rho_m\lambda_m)c(1-c)\Big(Kg + L\,\frac{\partial T}{\partial x}\Big)\frac{\partial T}{\partial x} = k_m\,\frac{\partial}{\partial x}\Big((1-Mc)\,\frac{\partial T}{\partial x}\Big),$$

where $g = |\boldsymbol{g}|$ and $f = f(t) = cu + (1-c)v$ is the volume averaged velocity of motion of the emulsion. To determine it, one needs to specify an additional boundary condition, for example, $f = 0$, for the case where the flow has a plane of symmetry or an impenetrable wall.

If $Kg = 0$ and $f(t) = \gamma/\sqrt{t}$, the one-dimensional problem admits the self-similar formulation

$$\begin{gathered} -\frac{\xi}{2}\,\frac{dc}{d\xi} + \Big((1-2c)L\,\frac{dT}{d\xi} - \gamma\Big)\frac{dc}{d\xi} + c(1-c)L\,\frac{d^2T}{d\xi^2} = 0, \\ \Big(-\frac{\xi}{2} - \gamma\Big)(\rho_d\lambda_d c + \rho_m\lambda_m(1-c))\,\frac{dT}{d\xi} + c(1-c)(\rho_d\lambda_d - \rho_m\lambda_m)\Big(L\,\frac{dT}{d\xi}\Big)^2 = \frac{d}{d\xi}\Big(k(c)\,\frac{dT}{d\xi}\Big), \end{gathered} \tag{1.2}$$

where $\xi = x/\sqrt{t}$, $c = c(\xi)$, and $T = T(\xi)$.

The most natural boundary-value problem for system (1.2) arises for the case $\gamma = 0$ under the boundary conditions

$$c(\infty) = c_\infty, \qquad T(0) = T_0, \qquad T(\infty) = T_\infty, \qquad T_0 > T_\infty \tag{1.3}$$

and corresponds to a sudden temperature rise on the left boundary. The less natural but simpler boundary-value problem

$$c(0) = c_0, \qquad LT_\xi(0) = \theta_0 \tag{1.4}$$

corresponds to the specification of a constant concentration different from the initial one and a temperature gradient of special form one the left boundary. We introduce the following notation: $\theta(\xi) = L\,dT/d\xi$. This quantity will be called the self-similar relative velocity [see (1.1)]. Everywhere below, $\gamma = 0$. System (1.2) can be written as

$$\begin{gathered} -\frac{\xi}{2}\,\frac{dc}{d\xi} + ((1-2c)\theta)\,\frac{dc}{d\xi} + c(1-c)\,\frac{d\theta}{d\xi} = 0, \\ -\frac{\xi}{2}\,(\rho_d\lambda_d c + \rho_m\lambda_m(1-c))\theta + c(1-c)(\rho_d\lambda_d - \rho_m\lambda_m)\theta^2 = \frac{d}{d\xi}\,(k(c)\theta). \end{gathered} \tag{1.5}$$

We linearize system (1.2) with respect to the low concentration, as was done in [3] for the problem of impurity solidification. As a result, the system breaks up, and the concentration is found from the equation

$$-\frac{\xi}{2}\,\frac{dc}{d\xi} + \frac{dc}{d\xi}\,\frac{dT}{d\xi} + c\,\frac{d^2T}{d\xi^2} = 0, \tag{1.6}$$

where $T(\xi)$ is a solution of the equation

$$-\frac{\xi}{2}\,\rho_m\lambda_m\,\frac{dT}{d\xi} = \frac{d}{d\xi}\Big(k_m\,\frac{dT}{d\xi}\Big). \tag{1.7}$$

The solution of problem (1.6), (1.7), (1.4) is written as

$$T(\xi) = T_0 + \theta_0 \int\limits_0^{\xi} \exp\Big(-\frac{\xi^2\rho_m\lambda_m}{4k_m}\Big)\,d\xi,$$
$$c(\xi) = c_0 \exp\Big(-\theta_0 \int\limits_0^{\xi} \frac{\xi\rho_m\lambda_m \exp(-\xi^2\rho_m\lambda_m/(4k_m))}{\xi k_m - 2k_m\theta_0 \exp(-\xi^2\rho_m\lambda_m/(4k_m))}\,d\xi\Big), \tag{1.8}$$

where

$$\theta_0 = (T_\infty - T_0)\Big/ \int\limits_0^{\infty} \exp(-\xi^2\rho_m\lambda_m/(4k_m))\,d\xi,$$
$$c_0 = c_\infty \exp\Big(\theta_0 \int\limits_0^{\infty} \frac{\xi\rho_m\lambda_m \exp(-\xi^2\rho_m\lambda_m/(4k_m))}{\xi k_m - 2k_m\theta_0 \exp(-\xi^2\rho_m\lambda_m/(4k_m))}\,d\xi\Big). \tag{1.9}$$

Unlike in the linearized problem, the existence of the solution of the complete self-similar problem (1.2), (1.3) is not obvious. Before proving the solvability of this problem, we consider the simpler auxiliary problem (1.5), (1.4).

2. Auxiliary Boundary-Value Problem. Solving system (1.5) with respect to the derivatives, we obtain the equations

$$\frac{d\theta}{d\xi} = \theta\Big(\frac{\xi}{2} - (1-2c)\theta\Big)\frac{-\xi(\rho_d\lambda_d c + \rho_m\lambda_m(1-c))/2 + c(1-c)(\rho_d\lambda_d - \rho_m\lambda_m)\theta}{\xi k(c)/2 - (k(c)(1-2c) - k'(c)c(1-c))\theta},$$
$$\frac{dc}{d\xi} = c(1-c)\theta\frac{-\xi(\rho_d\lambda_d c + \rho_m\lambda_m(1-c))/2 + c(1-c)(\rho_d\lambda_d - \rho_m\lambda_m)L\theta}{\xi k(c)/2 - (k(c)(1-2c) - k'(c)c(1-c))\theta}. \tag{2.1}$$

We introduce the functions

$$\psi(\xi, c, \theta) = \frac{\xi/2 - (1-2c)\theta}{\xi k(c)/2 - (k(c)(1-2c) - k'(c)c(1-c))\theta}; \tag{2.2}$$

$$\varphi(\xi, c, \theta) = \frac{-\xi(\rho_d\lambda_d c + \rho_m\lambda_m(1-c))/2 + c(1-c)(\rho_d\lambda_d - \rho_m\lambda_m)\theta}{\xi k(c)/2 - (k(c)(1-2c) - k'(c)c(1-c))\theta} \tag{2.3}$$

and write system (2.1) as

$$\frac{d\theta}{d\xi} = -\frac{\xi}{2}(\rho_d\lambda_d c + \rho_m\lambda_m(1-c))\theta\psi + c(1-c)(\rho_d\lambda_d - \rho_m\lambda_m)\theta^2\psi(\xi, c, \theta),$$
$$\frac{dc}{d\xi} = c(1-c)\theta\varphi(\xi, c, \theta). \tag{2.4}$$

For system (2.4) in the interval $[0, \infty)$, we consider the boundary-value problem (1.4), which becomes

$$c(0) = c_0, \qquad \theta(0) = \theta_0. \tag{2.5}$$

We seek the classical solution of problem (2.4), (2.5) in the interval $[0, \infty)$;

$$c \in [0, c^*], \tag{2.6}$$

where $c^* \leqslant 0.1$ is a positive number. It should be noted that constraint (2.6) does not complicate the problem since the examined model is constructed under the low concentration assumption.

The following result follows from the classical theorems for ordinary differential equations.

Lemma 2.1. *Problem* (2.4), (2.5) *with boundary conditions satisfying* (2.6) *has a unique solution which depends continuously on the initial conditions and is bounded in the interval* $[0, \infty)$.

Lemma 2.2. *For the solution of problem* (2.4), (2.5) *with boundary conditions satisfying* (2.6), *the following representations*

$$\theta(\xi) = \frac{\theta_0 \exp\Big(-\dfrac{1}{2}\displaystyle\int_0^\xi \psi\xi(\rho_d\lambda_d c + \rho_m\lambda_m(1-c))\,d\xi\Big)}{1 - \theta_0 \displaystyle\int_0^\xi c(1-c)(\rho_d\lambda_d - \rho_m\lambda_m)\psi \exp\Big(-\dfrac{1}{2}\int_0^\xi \psi\eta(\rho_d\lambda_d c + \rho_m\lambda_m(1-c))\,d\eta\Big)\,d\xi}; \tag{2.7}$$

$$c(\xi)(1 - c(\xi)) = c_0(1 - c_0)\exp\Big(\int_0^\xi (1-2c)\theta(\xi)\varphi(\xi, c, \theta)\,d\xi\Big) \tag{2.8}$$

and inequalities

$$\theta < 0, \qquad c > 0 \tag{2.9}$$

are valid.

Formulas (2.7)–(2.9) follow directly from Eqs. (2.4) and boundary conditions (2.5). The remaining properties of the solution $c(\xi)$ and $\theta(\xi)$ depend greatly on the sign of the difference $\rho_d\lambda_d - \rho_m\lambda_m$. Formulas (2.3), (2.7), and (2.8) imply that in the case $\rho_d\lambda_d - \rho_m\lambda_m \geq 0$, the functions $c(\xi)$ and $\theta(\xi)$ are monotonically increasing; if $\rho_d\lambda_d - \rho_m\lambda_m < 0$, they have minima in the interval considered, which slightly complicates the proof.

3. Estimates of the Self-Similar Relative Velocity. The following statement is checked directly.

Lemma 3.1. *If conditions* (2.6) *are satisfied, the function* ψ *given by formula* (2.2), *is a smooth function of its arguments and satisfies the inequality*

$$\alpha \leq \psi(\xi, c, \theta) \leq \beta, \tag{3.1}$$

where

$$\alpha = \frac{0{,}64}{\max(k_d, k_m)}, \qquad \beta = \frac{4}{\min(k_d, k_m)}$$

at $c \in [0, c^*]$, $c^* \leq 0.1$, *and* $P/\min(k_d, k_m) \leq 5$.

Lemma 3.2. *In the range of applicability of Lemma* 3.1, *the negative function* θ, *which is a solution of problem* (2.4), (2.5), *satisfies the inequalities*

$$|\theta_0|\exp(-\xi^2\beta\rho_d\lambda_d/4) \Big/ \Big(1 + |\theta_0|c^*\beta(\rho_d\lambda_d - \rho_m\lambda_m)\int_0^\infty \exp(-\xi^2\rho_m\lambda_m\alpha/4)\,d\xi\Big)$$

$$\leq |\theta| \leq |\theta_0|\exp(-\xi^2\alpha\rho_m\lambda_m/4) \tag{3.2}$$

in the case $\rho_d\lambda_d - \rho_m\lambda_m \geq 0$ *and*

$$|\theta_0|\exp(-\xi^2\rho_m\lambda_m\beta/4) \leq |\theta| \leq 2|\theta_0|\exp(-\xi^2\rho_d\lambda_d\alpha/4) \tag{3.3}$$

for

$$|\theta_0| < 1\Big/\Big(2\beta(\rho_m\lambda_m - \rho_d\lambda_d)c^*\int_0^\infty \exp(-\xi^2\rho_d\lambda_d\alpha/4)\,d\xi\Big) \tag{3.4}$$

in the case $\rho_d\lambda_d - \rho_m\lambda_m < 0$.

Lemma 3.2 is proved by using representation (2.7) for the function $\theta(\xi)$ and estimates (3.1) for the function ψ.

4. Estimates of Concentration. The following statement is checked directly.

Lemma 4.1. *In the case* $\rho_d\lambda_d - \rho_m\lambda_m \geq 0$, *the following relations are valid:*

$$\begin{gathered} -a \leq \varphi \leq 0, \\ a = \max\Big(\frac{\rho_m\lambda_d + (\rho_d\lambda_d - \rho_m\lambda_m)c^*}{\min(k_d, k_m)}, \frac{4c^*(\rho_d\lambda_d - \rho_m\lambda_m)}{\min(k_d, k_m)}\Big); \end{gathered} \tag{4.1}$$

in the case $\rho_d\lambda_d - \rho_m\lambda_m < 0$, the following relations are valid:

$$0 < \varphi(\xi) \le \frac{4c^*(\rho_m\lambda_m - \rho_d\lambda_d)}{\min(k_d, k_m)} \quad at \quad \xi \le \xi^*; \tag{4.2}$$

$$-\frac{\rho_m\lambda_m}{\min(k_d, k_m)} \le \varphi(\xi) \le 0 \quad at \quad \xi > \xi^*; \tag{4.3}$$

here ξ^* is a root of the equation $\varphi(\xi, c(\xi), \theta(\xi)) = 0$.

Lemma 4.1 is proved by using estimates (3.1) for the integral representations (2.7) and (2.8).

Lemma 4.2. *For the function $c(\xi)$ which is a solution of the auxiliary problem* (2.4), (2.5), *the following estimates hold:*

— *in the case $\rho_d\lambda_d - \rho_m\lambda_m \ge 0$,*

$$c_0(1-c_0) \le c(\xi)(1-c(\xi)) \le c_0(1-c_0)\exp\Big(|\theta_0| \int\limits_0^\xi a\exp(-\xi^2\alpha\rho_m\lambda_m/4)\,d\xi\Big); \tag{4.4}$$

— *in the case $\rho_d\lambda_d - \rho_m\lambda_m < 0$,*

$$c_0(1-c_0)\exp(-1) \le c(\xi)(1-c(\xi)) \le c_0(1-c_0)\exp\Big(\frac{\rho_m\lambda_m}{4(\rho_m\lambda_m - \rho_d\lambda_d)c^*}\Big). \tag{4.5}$$

Proof. To obtain a lower-bound estimate for the case of $\rho_d\lambda_d - \rho_m\lambda_m \ge 0$, it is sufficient to notice that the function $c(\xi)$ does not decrease monotonically, and to obtain an upper estimate, one needs to use representation (2.8), estimate (3.2) for the function θ, and estimate (4.1) for the function φ.

To obtain estimates for the case $\rho_d\lambda_d - \rho_m\lambda_d < 0$, we write representation (2.8) as

$$c(\xi)(1-c(\xi)) = c_0(1-c_0)\exp\Big(\int\limits_0^{\xi^*}(1-2c)\theta\varphi(\xi,c,\theta)\,d\xi + \int\limits_{\xi^*}^\infty(1-2c)\theta\varphi(\xi,c,\theta)\,d\xi\Big), \tag{4.6}$$

where ξ^* is a root of the equation $\varphi(\xi, c(\xi), \theta(\xi)) = 0$. Relation (4.6) implies the inequality

$$c(\xi)(1-c(\xi)) \ge c_0(1-c_0)\exp\Big(\int\limits_0^{\xi^*}(1-2c)\theta(\xi)\varphi(\xi,c,\theta)\,d\xi\Big),$$

which, together with estimates (3.3) and (4.2), leads to the lower estimate in (4.5). To obtain an upper estimate, we discard the first (negative) term in the argument of the exponent in equality (4.6). As a result, we obtain the inequality

$$c(\xi)(1-c(\xi)) \le c_0(1-c_0)\exp\Big(\int\limits_{\xi^*}^\infty(1-2c)\theta\varphi(\xi,c,\theta)\,d\xi\Big).$$

Thus, in view of (3.3) and (4.3), inequality (4.5) and Lemma 4.2 are proved.

5. Existence of a Self-Similar Solution of the Primal Problem. We return to problem (1.2), (1.3) in the case $\gamma = 0$. We integrate (2.7) in the interval from 0 to ∞ and substitute $\xi = \infty$ into formula (2.8). As a result, we have the equations

$$T_\infty - T_0 = \frac{1}{L}\int\limits_0^\infty \theta\,d\xi \equiv -F(\theta_0, c_0),$$

$$c_\infty(1-c_\infty) = c_0(1-c_0)\exp\Big(\int\limits_0^\infty(1-2c)\theta(\xi)\varphi(\xi,c,\theta)\,d\eta\Big) \equiv G(\theta_0, c_0).$$

Let us consider the case $\rho_d\lambda_d - \rho_m\lambda_m \geq 0$. Estimates (3.2) imply the inequality

$$|\theta_0| \int\limits_0^\infty \exp(-\xi^2\beta\rho_d\lambda_d/4)\,d\xi \Big/ \Big[L\Big(1 + |\theta_0|c^*\beta(\rho_d\lambda_d - \rho_m\lambda_m)\int\limits_0^\infty \exp(-\xi^2\rho_m\lambda_m\alpha/4)\,d\xi\Big)\Big]$$

$$\leq F(\theta_0, c_0) \leq \frac{|\theta_0|}{L}\int\limits_0^\infty \exp(-\xi^2\alpha\rho_m\lambda_m/4)\,d\xi,$$

from which, by virtue of the continuity of F and the conditions $F(0, c_0) = 0$, it follows, that for any T_0 and T_∞ satisfying the conditions

$$0 < T_0 - T_\infty \leq \frac{\sqrt{\rho_m\lambda_m\alpha}}{\sqrt{\rho_d\lambda_d\beta}\,L\beta c^*(\rho_d\lambda_d - \rho_m\lambda_m)}, \tag{5.1}$$

and for any $c_0 \in (0, c^*]$, there exists $\theta_0 < 0$ such that

$$T_0 - T_\infty = F(\theta_0, c_0). \tag{5.2}$$

We denote by $\theta_0(c_0)$ the smallest (in modulus) solution of Eq. (5.2).

Using estimate (4.4), we obtain

$$c_0(1 - c_0) \leq G(\theta_0(c_0), c_0) \leq c_0(1 - c_0)\exp\Big(|\theta_0(c_0)|\int\limits_0^\infty a\exp(-\xi^2\alpha\rho_m\lambda_m/4)\,d\xi\Big).$$

Because the function G is continuous and vanishes for $c_0 = 0$, it follows that for any c_∞ satisfying the conditions

$$0 < c_\infty \leq c^* \tag{5.3}$$

there exists a value $c_0 \in (0, c_\infty)$ such that the equation

$$c_\infty(1 - c_\infty) = G(\theta_0(c_0), c_0) \tag{5.4}$$

has a solution. Thus, the existence of a solution of system (2.1) for the initial data of the problem satisfying conditions (5.1) and (5.3) is proved. The temperature distribution is found from the formula

$$T(\xi) = \int\limits_0^\xi \theta\,d\xi + T_0. \tag{5.5}$$

Hence, the following theorem is proved.

Theorem 1. *Let $\rho_d\lambda_d - \rho_m\lambda_m \geq 0$. Then, for any T_0, T_∞, and c_0 satisfying conditions* (5.1) *and* (5.3), *the following relations are valid:*

$$0 < T_0 - T_\infty \leq \frac{\sqrt{\rho_m\lambda_m\alpha}}{\sqrt{\rho_d\lambda_d\beta}\,L\beta c^*(\rho_d\lambda_d - \rho_m\lambda_m)}, \qquad 0 < c_\infty \leq c^*;$$

here c^* is any number in the interval $(0, 0.1)$. *In this case, problem* (1.5) *has a classical solution bounded over the entire interval* $[0, \infty)$.

Let us consider the case $\rho_d\lambda_d - \rho_m\lambda_m < 0$. Inequality (3.3) of Lemma 3.2 leads to the estimate

$$\frac{|\theta_0|}{L}\int\limits_0^\infty \exp(-\xi^2\beta\rho_m\lambda_m/4)\,d\xi \leq F(\theta_0, c_0) \leq \frac{2|\theta_0|}{L}\int\limits_0^\infty \exp(-\xi^2\alpha\rho_d\lambda_d/4)\,d\xi.$$

In view of constraints (3.4) on the value of θ_0 and the continuity condition for the function $F(\theta_0, c_0)$, it is concluded that for any T_0 and T_∞ satisfying the conditions

$$0 < T_0 - T_\infty \leq \frac{1}{L\beta c^*(\rho_m\lambda_m - \rho_d\lambda_d)}, \tag{5.6}$$

and for any $c_0 \in (0, c^*]$, there exists a value $\theta_0 < 0$ that is a solution of Eq. (5.2). As above, by $\theta_0(c_0)$ we denote the smallest (in modulus) solution of Eq. (5.2).

Estimate (4.5) of Lemma 4.2 leads to

$$c_0(1-c_0)\exp(-1) \le G(\theta_0(c_0), c_0) \le c_0(1-c_0)\exp\Big(\frac{\rho_m\lambda_m}{4(\rho_m\lambda_m-\rho_d\lambda_d)}\,c^*\Big).$$

From this, we conclude that for any c_∞ satisfying the conditions

$$0 < c_\infty(1-c_\infty) \le c^*(1-c^*)\exp(-1), \tag{5.7}$$

there exists a value $c_0 \in (0, c_\infty)$ such that Eq. (5.4) has a solution. The temperature is found from formula (5.5). Thus, the following theorem is proved.

Theorem 2. *Let* $\rho_d\lambda_d - \rho_m\lambda_m < 0$. *Then, for any* T_0, T_∞, *and* c_0 *satisfying conditions* (5.6) *and* (5.7), *the following relations are valid:*

$$0 < T_0 - T_\infty \le \frac{1}{L\beta c^*(\rho_m\lambda_m - \rho_d\lambda_d)}, \qquad 0 < c_\infty(1-c_\infty) \le c^*(1-c^*)\exp(-1);$$

here c^* *is any number in the interval* $(0, 0.1)$. *In this case, problem* (1.5), (1.4) *has the classical solution bounded over the entire the range* $[0,\infty)$.

Remark 1. In Theorems 1 and 2, the constraints imposed on the initial data can be refined but in the form presented here they also do not complicate the problem and hold in most real cases, as follows from the examples considered below. In the case $\rho_d\lambda_d = \rho_m\lambda_m$, there are no upper-bound constraints on the difference $T_0 - T_\infty$.

6. Examples of Numerical Calculations. Since the numerical solution of the problem of motion of an emulsion is a subject of separate research, we restrict ourselves to examples of numerical solution of the Cauchy problem for the system of ordinary differential equations (of the auxiliary problem) obtained using the MathCad software. We consider two typical cases corresponding to the conditions of Theorems 1 and 2. The constants characterizing the thermal properties of the elements are taken from [4]; the value of R was set equal to 10^{-5} m. The results corresponding to the conditions of Theorems 1 and 2 are given below. In particular, in the examples (see below), the auxiliary problem was solved subject to the conditions $c_0 = 0.001$ and $\theta_0 = -0.025$ m/sec$^{1/2}$.

Example 1. The carrier phase is lead, and the dispersed phase is aluminum. The values of the parameters $\rho_d\lambda_d = 2{,}584{,}911$ J/(K $\cdot$ m^3), $\rho_m\lambda_m = 1{,}431{,}855$ J/(K $\cdot$ m^3), $k_d = 62$ J/(K $\cdot$ m $\cdot$ sec), $k_m = 19.77$ J/(K $\cdot$ m $\cdot$ sec), $L = 0.00000608$m^2/(K $\cdot$ sec) [4] correspond to the condition of Theorem 1. In Fig. 1, this solution corresponds to the monotonic concentration profile which reaches the asymptote $c = 0.0062028163$ (curve 1). The temperature difference was 27.442 K. In the case considered, $c^* \le 0.007$ and condition (5.1) is satisfied.

The solution of the problem linearized in the low impurity concentration [formulas (1.8)] corresponds to curve 2 in Fig. 1. Differences are observed only for very small arguments (of the order of 10^{-4}). However, even in the scale of Fig. 1, it is evident that the solution of the linearized problems reaches an asymptote that is slightly below the asymptote of the solutions of the complete problem ($c = 0.0061825704$).

Example 2. The carrier phase is aluminum, and the dispersed phase is lead. The values of the parameters $\rho_d\lambda_d = 1{,}431{,}855$ J/(K $\cdot$ m^3), $\rho_m\lambda_m = 2{,}584{,}911$ J/(K $\cdot$ m^3), $k_d = 19.77$ J/(K $\cdot$ m $\cdot$ sec), $k_m = 62$ J/(K $\cdot$ m $\cdot$ sec), and $L = 0.00000568$ m^2/(K $\cdot$ sec) [4] correspond to the condition of Theorem 2.

In this case, the asymptotic value is $c = 0.005116193$ and the temperature difference is 38.206 K. From the calculation results it follows that in the vicinity of zero, the concentration changes nonmonotonically ($c_{\min} = 0.0009999948$). In this case, $c^* \le 0.006$ and conditions (5.6) and (5.7) and constraint (3.4) are satisfied. As in example 1, the solution of the problem linearized in the low impurity concentration is given, to which curve 3 in Fig. 1 corresponds. The solution of the linearized problems is strictly monotonic and reaches an asymptote which is slightly below the asymptote of the solutions of the complete problem ($c = 0.0051252821$), to which curve 4 in Fig. 1 corresponds.

Example 3. Two emulsions under the same conditions (1.3). Since the concentration profiles in the complete and linearized problems are practically indistinguishable in the scale of Fig. 1, we examine the initial-boundary-value problem (1.3) for two emulsions for the same initial data ($c_\infty = 0.01$, $T_\infty = 1173$ K, and $T_0 = 1153$) by calculating the corresponding values of θ_0 and c_0 using formulas (1.9). For the lead–aluminum emulsion, $c_0 = 0.002$ and $\theta_0 = -0.01822$ m/sec$^{1/2}$, and for the aluminum–lead emulsion, $c_0 = 0.0029206$ and $\theta_0 = -0.013$ m/sec$^{1/2}$.

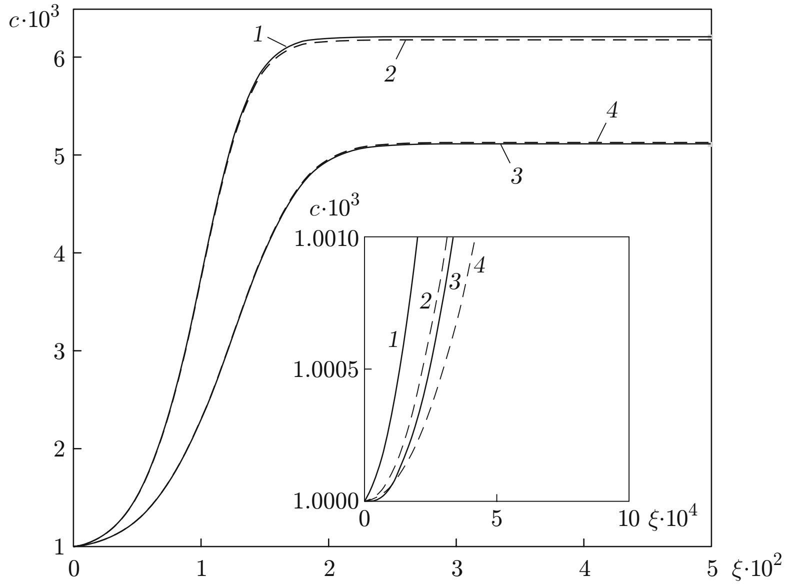

Fig. 1. Concentration distribution in the linearized (1 and 3) and complete (2 and 4) problems: curves 1 and 2 refer to the aluminum concentration in a lead–aluminum emulsion and curves 3 and 4 refer to the lead concentration in an aluminum–lead emulsion.

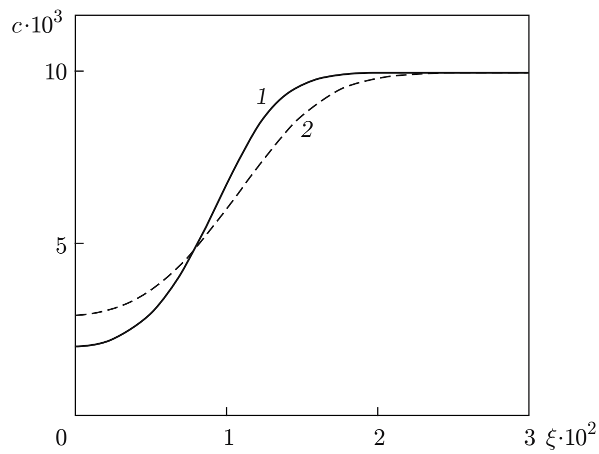

Fig. 2. Concentration distribution in the linearized problems for the same initial data: curve 1 refers to the aluminum concentration in a lead–aluminum emulsion and curve 2 refers to the lead concentration in an aluminum–lead emulsion.

Figure 2 shows curves of $c(\xi)$ for the two emulsions (curve 1 is the concentration profile in the linearized problem for the lead–aluminum emulsion, and curve 2 is the same for the aluminum–lead emulsion). In this scale, the plots of the solution of the complete problems differ little from the corresponding plots of the solutions of the linearized problems. Differences are observed for higher concentrations, for example, $c = 0.05$.

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